A universal pattern of the evolution of life table entropy and life expectancy

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Short Abstract

Using the complete data collection of the Human Mortality Database (HMD), we demonstrate a striking and heretofore-undocumented regularity in the relationship between life expectancy, $e(0)$, and Shannon entropy, H. We show that $\sqrt{H}$ is a 2-segment picewise-linear function of $e(0)$ when the two quantities are considered in historical terms — viz., plotted over time for a given country. Entropy has often been considered as a candidate for a measure of life table rectangularization, but our work suggests that in real human life tables, entropy contains little information beyond that already conveyed by $e(0)$. This does not imply that H is not of interest theoretically, but it does suggest that as an empirical matter, demographers should look elsewhere in their quest to find a quantity that measures rectangularization. Additionally, we explore the interpretation of the 2-segment cut-point in light of prior work on the modal age of death.

1 Background

Life tables and life expectancy

A life table is a nonparametric statistical tool used by demographers to analyze mortality. Life tables have several functions (or 'columns'); relevant here is $\ell(x)$ which is the survivorship function at age $0 \leq x \leq \omega$, where ω is some hypothetical age beyond which no one is presumed to live. Practical applications often use $\omega = 115$ or similar. The life table survivorship function is a monotone decreasing function: $\ell(0) \equiv 1 \geq \ell(x) \geq 0 \equiv \ell(\omega)$; in short, $\ell(x)$ is the proportion surviving to age x, calculated from mortality rates in a given period (using death reports and population estimates). The ubiquitous statistic 'life expectancy', or, in the full demography-jargon, 'the complete expectation of life at birth' is $e(0)$, where:

$$e(x) = \int_x^{\omega} \ell(a)da.$$

Although demographers define life expectancy at any age x, in everyday parlance, 'life expectancy' is $e(0)$. This is standard demography (see, e.g., Keyfitz 1970).

Life table entropy

Demetrius (1974) adapted entropy, H (Shannon, 1948), to the life table $\ell(x)$ column:

$$H(x) = \frac{-\int_x^{\omega} \log\left[\ell(a)\right] \ell(a)da}{\int_x^{\omega} \ell(a)da}.$$

As a shorthand, $H \equiv H(0)$.

Life table entropy is a measure of the rectangularity of the $\ell(x)$ vs x curve. In a fully rectangular life table, there are no deaths until age ω, when everyone dies; in such a table $e(0) = \omega$ and $H = 0$. If deaths are exponentially-distributed, or $\ell(x) = \exp(-\mu x)$ where μ is a constant force of mortality, then $H = 1$; exponential life tables are considered extreme for humans, but have been used (see, e.g., McLean and Anderson 1988). Exponential survivorship occurs in some non-human species (e.g. barnacles and oysters, see Deevey 1947; Wilson and Bossert 1971). An $\ell(x)$ function that is a straight diagonal line has $H = 0.5$; such life tables occur in passeriform birds (Deevey 1947). *Hydra spp.* have been alternately reported to have linear (Pearl and Miner 1935) and exponential (Stiven 1962; Martínez 1998) survivorship . Keyfitz (1985) calls H "a convenient summary of the degree of concavity in an $\ell(x)$ column" (p. 64).

2 Rectangularization

As life expectancy increases, the life table $\ell(x)$ curve becomes more rectangular. This had spurred interest in the phenomenon of 'rectangularization' (Cheung et al., 2005) as a measure of mortality, vis-à-vis population aging. In principle, life expectancy can increase by *any* changes to mortality rates which make $\int_0^{\omega} \ell(a)da$ larger (including expansion of ω), subject to the $\ell(x)$ monotonicity constraint. In practice, ω changes rather slowly over time (122 years being the current value, Robine and Allard 1999), so most of the recent evolution of $e(0)$ is indeed rectangularization in the strict sense.

3 Our findings

Apart from some heuristic or theoretical work (see e.g. Vaupel 1986; Goldman and Lord 1986), there has been little work on entropy and life expectancy. There is even less empirical work. Nagnur (1986) looked at $H(x)$ for various ages in Canada, and Wilmoth and Horiuchi (1999) did a round-robin comparison of rectangularization measures including H, using real life tables.

We performed the most intensive empirical investigation of H in real human life tables of which we are aware, calculating $e(0)$ and H for all 40 countries in the Human Mortality Database (HMD 2012), comprising 3,671 life tables in total.

We find a remarkably linear relationship between $e(0)$ and $\sqrt{H}$. Figure 1 plots all the data, which shows a striking uniformity among the HMD countries in the way in which Shannon Entropy declines as life expectancy increases. There are a number of notable features in figure 1:

1. Strikingly linear relationship between H and $e(0)$, once H has been square-root transformed.
   - This could suggest an intriguing universality whereby real human life tables always follow a similar path to long $e(0)$ in terms of the shape of the $\ell(x)$ curve.
   - A more pedestrian explanation is that for human life tables, Shannon entropy is simply not very informative. It changes, but — modulo changes in $e(0)$ — it does so in a predictable way.
2. Straight-line is a first-order approximation. There is a clear, if subtle, 'hockey-stick' pattern of $\sqrt{H}$ being picewise-linear in figure 1, with the elbow at $e(0) \approx 66$. There is some relatively recent literature [we are still investigating] suggesting that this is a pivot level of $e(0)$ with respect to the SD of life table ages of death; probably a related phenomenon.

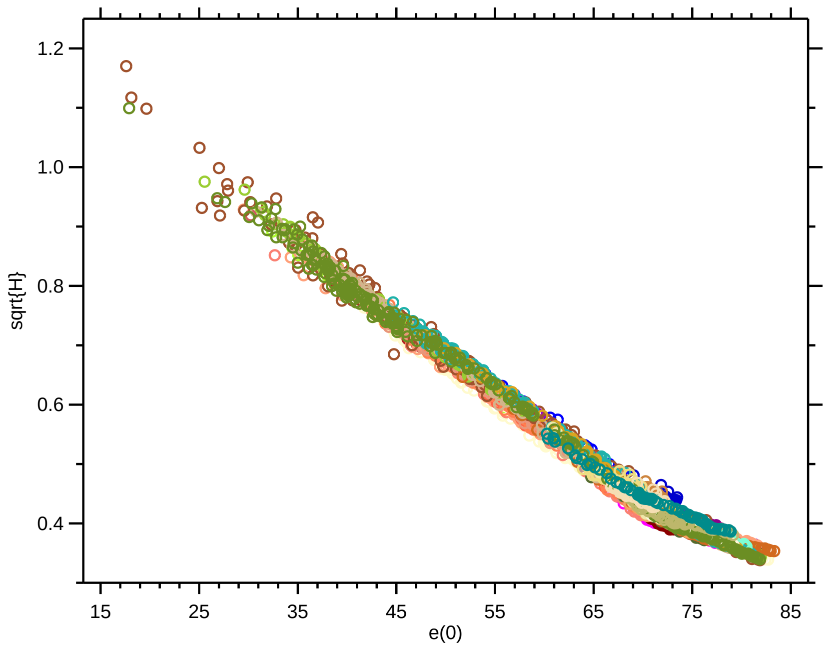

Figure 1: Square root of Shannon entropy ($\sqrt{H}$), vs life expectancy ($e(0)$), for 3,671 life tables in the HMD. Each country is plotted in a different color.

3. It has been suggested that $H \lesssim 1.0$ in human populations. While this is not a very important point, it is clearly falsified, since several empirical human life tables in the HMD satisfy $H > 1.0$.

4 Next step

Figures 2 and 3 present the same data as figure 1, disaggregated by country. The picewise-linearity noted in point 2 above is apparent for many countries.

We have fit picewise-linear per-country functions, with endogenous cutpoints. The cutpoints are illustrated by the vertical rules in the graph.

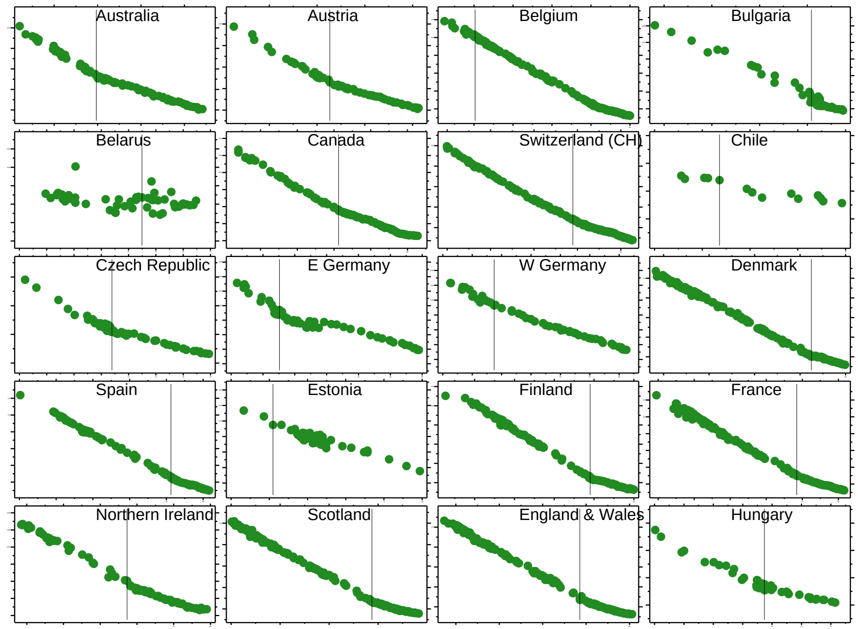

Figure 2: Vertical rules denote cut point in picewise-linear function. Continues in figure 3.

5 Conclusion

Life table Shannon entropy, H, is not a useful measure of rectangularization.

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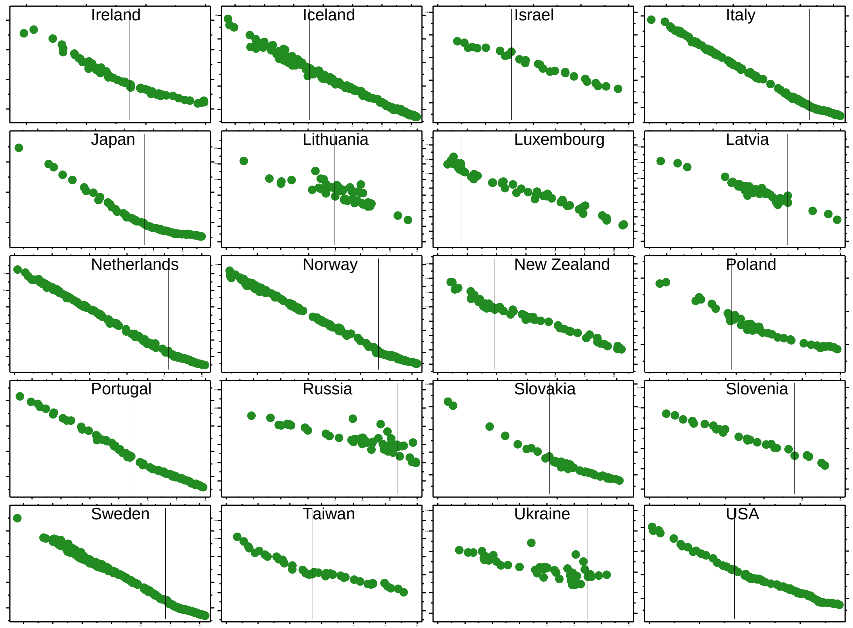

Figure 3: Continuation of figure 2.

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